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1 Invariant Subspace

Definition 1.1. Let T be a linear operator on a vector space V. A subspace W of V is called a *T-invariant subspace* of V if $T(W) \subseteq W$, i.e., for all $w \in W$, we have $T(w) \in W$.

Examples to keep in mind:

For any linear operator T on V

- $\{0\}$ is T-invariant;
- V is T-invariant;
- $\operatorname{im} T$ is T-invariant;
- $\ker T$ is T-invariant;
- E_λ for any eigenvalue λ of T is T-invariant.

Definition 1.2. Let T be a linear operator on V, and let x be a nonzero vector in V. The subspace

$$W = \operatorname{span}\{x, T(x), T^2(x), ...\}$$

is called the *T-cyclic subspace of V generated by x*.

Useful Facts

Let T be a linear operator on a finite-dimensional vector space V and let W denote the T-cyclic subspace of V generated by a nonzero vector $v \in V$. Let $k = \dim W$. Then

(a) $\{v, T(v), T^2(v), ..., T^{k-1}(v)\}$ is a basis for W.

(b) If $a_0 v + a_1 T(v) + \cdots + a_{k-1} T^{k-1}(v) + T^k(v) = 0$, then the characteristic polynomial of T_W is $f(t) = (-1)^k \left(a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k\right)$.

Exercise:

Q1

Let $W \subseteq V$ be T-invariant and $\lambda_1, ..., \lambda_k$ be distinct eigenvalues of T. Suppose there are $v_1, ..., v_k \in V$ such that $v_i \in E_{\lambda_i}(T)$ and $v_1, ..., v_k \in W$. Show that $v_i \in W$ for all i.

Solution:
We prove it by induction. Suppose $v_1 \in V$ and $v_1 \in W$, then the result is obviously true. Suppose that $k = n - 1$ is true. Then for $k = n$, since W is T-invariant, we have

$$v_1 + \cdots + v_n \in W \implies T(v_1) + \cdots + T(v_n) \in W$$

moreover, since $v_i \in E_{\lambda_i}(T)$ for all $i = 1, ..., n$, we have

$$T(v_1 + \cdots + v_{n-1} + v_n) - \lambda_n(v_1 + \cdots + v_{n-1} + v_n) = (\lambda_1 - \lambda_n)v_1 + \cdots + (\lambda_{n-1} - \lambda_n)v_{n-1} \in W.$$

Moreover, each $(\lambda_i - \lambda_n)v_i \in E_{\lambda_i}(T)$ because they are distinct, hence the induction hypothesis implies that each $(\lambda_i - \lambda_n)v_i$ is in W, for $i = 1, ..., n - 1$ so that $v_i \in W$ for each $i = 1, ..., n - 1$. Hence,

$$v_n = (v_1 + \cdots + v_n) - (v_1 + \cdots + v_{n-1}) \in W.$$

Q2

Let T be a linear operator on V and let $W_1, ..., W_k$ be T-invariant subspaces of V. Prove that $W_1 + \cdots + W_k$ is also a T-invariant subspace of V.

Solution:
Let $W := W_1 + \cdots W_k$. Then for all $w \in W$, we can write

$$w = w_1 + \cdots + w_k$$

for some $w_1 \in W_1, ..., w_k \in W_k$. Since each W_i is T-invariant, we have

$$T(w) = T(w_1 + \cdots + w_k) = T(w_1) + \cdots + T(w_k) \in W_1 + \cdots + W_k = W$$

so that W is also T-invariant.

2 Cayley-Hamilton Theorem

Theorem 2.1 (Cayley-Hamilton Theorem). *Let T be a linear operator on a finite-dimensional vector space V, and let $f(t)$ be the characteristic polynomial of T. Then $f(T) = T_0$, i.e., the zero transformation.*

In other words, if you evaluate f at "T", then it returns "0" – as a linear transformation.

Corollary 2.1 (Caley-Hamilton Theorem; Matrix version). *Let A be an $n \times n$ matrix, and let $f(t)$ be the characteristic polynomial of A. Then $f(A) = O$, i.e., the $n \times n$ zero matrix.*

Exercises

Q3: Computation

Let $A = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$.

(a) Show that the Cayley-Hamilton theorem does hold in this case.

(b) Suppose that $at + b$ is the remainder of t^n when divided by $f_A(t) := \det(A - tI)$. Find the values of a and b.

(c) Using (a) and (b), compute A^n.

Solution:

(a) We have that

$$f_A(A) := \det(A - tI) = (-1 - t)(-t) - 2 = t^2 + t - 2 = (t - 1)(t + 2)$$

then

$$f_A(A) = (A - I)(A + 2I) = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(b) Given that $t^n = f_A(t)q(t) + at + b$, where $q(t)$ is some polynomial and $a, b \in \mathbb{R}$. Let $t = 1$ and $t = -2$, we have

$$1 = a + b$$

and

$$(-2)^n = -2a + b$$

which yields

$$a = \frac{1 - (-2)^n}{3} \text{ and } b = \frac{2 + (-2)^n}{3}$$

(c) Then $A^n = f_A(A)q(A) + aA + bI$, by Cayley-Hamilton, we have

$$A^n = aA + bI = \frac{1}{3} \begin{pmatrix} 1 - (-2)^{n+1} & 2 + (-2)^{n+1} \\ 1 - (-2)^n & 2 + (-2)^n \end{pmatrix}$$

Q4

Let A be an $n \times n$ matrix which has the following characteristic polynomial:

$$f_A(t) = t^n - a_{n-1}t^{n-1} + \cdots + (-1)^{n-1}a_1 t + (-1)^n a_0.$$

Show that $a_{n-1} = \operatorname{tr} A$ and $a_0 = \det A$. Furthermore, deduce that A is invertible if and only if $a_0 \neq 0$.

Solution:
The t^{n-1}-term in $(a_{11} - t)(a_{22} - t) \cdots (a_{nn} - t)$ is precisely

$$-(a_{11} + a_{22} + \cdots + a_{nn}) = -\operatorname{tr} A.$$

While

$$f_A(t) = \det(A - tI)$$

and $f_A(0) = a_0 = \det A$.

Q5

Suppose that A is invertible, then

$$A^{-1} = -\frac{1}{\det A}\left[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n\right].$$

Show, if A is an invertible 2×2 matrix, then

$$A^{-1} = \frac{1}{\det A}\left[(\operatorname{tr} A)I - A\right].$$

Solution:
Given

$$f_A(t) = t^n - a_{n-1}t^{n-1} + \cdots + (-1)^{n-1}a_1 t + (-1)^n a_0$$

we have

$$0 = A^n - a_{n-1}A^{n-1} + \cdots + (-1)^{n-1}a_1 A + (-1)^n a_0 I$$

multiplying both sides with A^{-1} yields

$$(-1)^{n+1}a_0 A^{-1} = A^{n-1} - a_{n-1}A^{n-2} + \cdots + (-1)^{n-1}a_1 I$$

hence

$$A^{-1} = -\frac{1}{\det A}\left[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n\right].$$

For the 2×2 case, we simply writes $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ or use the above formula to deduce.

3 Recordings

Tuesday
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Thursday
Link: Here
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